ADVANCED NUMBER THEORY FINAL EXAM

This exam is of **50 marks** and is **4 hours long** - from 10 am to 2pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may also refer to the books by

- K. Ireland and M. Rosen - **A Classical Introduction to Modern Number Theory**.
- N. Koblitz - **An Introduction to Elliptic Curves and Modular Forms** .
- J-P Serre - **A Course in Arithmetic**.

If you have any questions please call me at **+91 98804 59642** or email me at **rameshsreekantan@gmail.com**.
Please sign the following statement and scan this sheet along with the rest.

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Consider the quadratic equation

$$F(X) = X^2 + X + 1 = 0$$

Let P_F be the set of primes such that $F(X) = 0$ has a root in $\mathbb{F}_p$. Show that P_F has density 1/2. 7

b. What is the density of primes for which the equation

$$G(X) = (X^2 - 11)(X^2 - 187)(X^2 - 17)$$

has a root? 3

2. Compute the Zeta function of the equation given by the affine equation

$$E : Y^2 = X^3 + X^2$$

Hint: One way to do this is to projectivise and count. Alternately, you can count the projective and affine solutions separately. 10

3a. Compute the dimension of the space of cusp forms of weight 48 for $\Gamma = SL_2(\mathbb{Z})$ 5

3b. Explicity write down a basis for this space in terms of Eisenstein series. 5

4. Let $E_k(z)$ be the Eisenstein series of weight k

$$E_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^k}$$

Show that $E_4(z)$ satisfies the differential equation

$$\frac{3}{2\pi i} F'(z) - E_2(z)F(z) + E_6(z) = 0$$

where $E_2(z)$ is the Eisenstein series of weight 2. 7

4b. Use this to express σ_5 in terms of σ_3 and σ_1. 3

5a. Let p be a prime and $d = (m, p-1)$. Show that the number of solutions $N(X^m = a)$ in the field $\mathbb{F}_p$ is

$$N(X^m = a) = \sum_{\chi} \chi(a)$$

where χ runs through characters of order d. 5

5b. Count the number of solutions of

$$X^3 - Y^3 = 1$$

in $\mathbb{F}_{19}$. 5